Exercise 4

A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.

(a) What is the relative growth rate? Express your answer as a percentage.

(b) What was the intitial size of the culture?

(c) Find an expression for the number of bacteria after t hours.

(d) Find the number of cells after 4.5 hours.

(e) Find the rate of growth after 4.5 hours.

(f) When will the population reach 50,000?

[TYPO: This should be spelled as "initial."]

Solution

Part (a)

Assume that the relative growth rate is constant.

$$\frac{1}{P}\frac{dP}{dt} = k$$

Rewrite the left side using the chain rule.

$$\frac{d}{dt}\ln P = k$$

The function that you take a derivative of to get k is $kt + C$, where C is any constant.

$$\ln P = kt + C$$

Exponentiate both sides to get P.

$$\begin{aligned} P(t) &= e^{kt+C} \\ &= e^C e^{kt} \end{aligned}$$

Use a new constant P_0 for e^C.

$$P(t) = P_0 e^{kt} \tag{1}$$

The bacteria count was 400 after 2 hours and 25,600 after 6 hours. Use this information to construct a system of equations for the two unknowns, P_0 and k.

$$\begin{cases} P(2) = P_0 e^{2k} = 400 \\ P(6) = P_0 e^{6k} = 25\,600 \end{cases}$$

In order to eliminate P_0, divide both sides of the second equation by those of the first equation.

$$\frac{P_0 e^{6k}}{P_0 e^{2k}} = \frac{25\,600}{400}$$

$$e^{4k} = 64$$

$$\ln e^{4k} = \ln 64$$

$$4k = \ln 64$$

$$k = \frac{\ln 64}{4} \approx 1.03972 \text{ hour}^{-1}$$

Therefore, the relative growth rate is about 104% per hour; that is, the population increases by a factor of about 2.04 every hour.

Part (b)

Substitute this value of k into either of the two equations to get P_0.

$$P_0 e^{2k} = 400$$

$$P_0 e^{2\left(\frac{\ln 64}{4}\right)} = 400$$

$$P_0 e^{\frac{1}{2}\ln 64} = 400$$

$$P_0 e^{\ln 64^{1/2}} = 400$$

$$P_0 \sqrt{64} = 400$$

$$P_0(8) = 400$$

$$P_0 = 50$$

The initial population of bacteria was 50.

Part (c)

Plugging the values for k and P_0 into equation (1), the bacteria population at t hours is therefore

$$\begin{aligned} P(t) &= 50 e^{\left(\frac{\ln 64}{4}\right)t} \\ &= 50 e^{\ln 64^{t/4}} \\ &= 50(64)^{t/4}. \end{aligned}$$

Part (d)

After 4.5 hours, the population is

$$P(4.5) = 50(64)^{4.5/4} \approx 5382.$$

Part (e)

After 4.5 hours the rate of population growth is

$$\left.\frac{dP}{dt}\right|_{t=4.5} = kP(4.5) = \left(\frac{\ln 64}{4}\right)\left[50(64)^{4.5/4}\right] \approx 5595.5\ \frac{\text{cells}}{\text{hour}}.$$

Part (f)

To find when the population will reach 50,000, set $P(t) = 50{,}000$ and solve the equation for t.

$$P(t) = 50\,000$$

$$50(64)^{t/4} = 50\,000$$

$$64^{t/4} = 1000$$

$$\ln 64^{t/4} = \ln 1000$$

$$\frac{t}{4}(\ln 64) = \ln 1000$$

$$t = \frac{4 \ln 1000}{\ln 64} \approx 6.64386 \text{ hours}$$